

ON THE HIERARCHICAL RISK-AVERSE CONTROL PROBLEMS FOR DIFFUSION PROCESSES *

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Abstract. In this paper, we consider a risk-averse control problem for diffusion processes, in which there is a partition of the admissible control strategy into two decision-making groups (namely, the *leader* and *follower*) with different cost functionals and risk-averse satisfactions. Our approach, based on a hierarchical optimization framework, requires that a certain level of risk-averse satisfaction be achieved for the *leader* as a priority over that of the *follower's* risk-averseness. In particular, we formulate such a risk-averse control problem using partially coupled *forward-backward stochastic differential equations* that allow us to introduce a family of time-consistent dynamic convex risk measures, based on backward-semigroup operators, w.r.t. the strategies of the *leader* and that of the *follower*. Moreover, under suitable conditions, we establish the existence of optimal risk-averse solutions, in the sense of viscosity solutions, to the associated risk-averse dynamic programming equations. Finally, we remark on the implication of our result in assessing the influence of the *leader's* risk-averse satisfaction on the risk-averseness of the *follower* in relation to the direction of *leader-follower* information flow.

Key words. Dynamic programming equation, forward-backward SDEs, hierarchical control, risk-averse control, value functions, viscosity solutions.

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1. Introduction. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a probability space, and let $\{B_t\}_{t \geq 0}$ be a d -dimensional standard Brownian motion, whose natural filtration, augmented by all \mathbb{P} -null sets, is denoted by $\{\mathcal{F}_t\}_{t \geq 0}$, so that it satisfies the *usual hypotheses* (e.g., see [23] or [11]). We consider the following controlled-diffusion process over a given finite-time horizon $T > 0$

$$dX_t = f(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dB_t, \quad X_0 = x, \quad 0 \leq t \leq T, \quad (1.1)$$

where

- X is an \mathbb{R}^d -valued diffusion process,
- u is a U -valued measurable control process (i.e., an admissible control from some measurable set $\mathcal{U} \subset U$, where U is a compact set in \mathbb{R}^d) such that for all $t > s$, $(B_t - B_s)$ is independent of u_r for $r \leq s$ (nonanticipativity condition) and

$$\mathbb{E} \int_s^t |u_\tau|^2 d\tau < \infty, \quad \forall t \geq s,$$

- the function $f: [0, T] \times \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ is uniformly Lipschitz, with bounded first derivative, and

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- $\sigma: [0, T] \times \mathbb{R}^d \times U \rightarrow \mathbb{R}^{d \times d}$ is Lipschitz with the least eigenvalue of $\sigma \sigma^T$ uniformly bounded away from zero, i.e.,

$$\sigma(t, x, u) \sigma^T(t, x, u) \succeq \lambda I_{d \times d}, \quad \forall (x, u) \in \mathbb{R}^d \times U, \quad \forall t \in [0, T],$$

for some $\lambda > 0$.

In this paper, we specifically consider a hierarchical risk-averse control problem for the above controlled-diffusion process, in which there is a partition of the admissible control strategy into two decision making groups (i.e., progressively measurable strategies corresponding to the *leader* and *follower* – where such notions are used in the Stackelberg’s optimization [27]) with different cost functionals and risk-averse satisfactions. In particular, we partition the control subdomain U into two open sets V and W (with $V \cap W = \emptyset$) that are compatible with the strategy subspaces of the *leader* and that of the *follower*, respectively. That is,

$$U = V \cup W \text{ up to a set of measurable } U, \quad (1.2)$$

where the risk-averse strategy for the *leader* v is a V -valued measurable control process from the set $\mathcal{V}_{[0, T]}$ with

$$\begin{aligned} \mathcal{V}_{[0, T]} \triangleq \left\{ v: [0, T] \times \Omega \rightarrow V \mid v \text{ is an } \{\mathcal{F}_t\}_{t \geq 0} \text{-adapted} \right. \\ \left. \text{and } \mathbb{E} \int_0^T |v_t|^2 dt < \infty \right\} \end{aligned} \quad (1.3)$$

and the risk-averse strategy for the *follower* w is a W -valued measurable control process from the set $\mathcal{W}_{[0, T]}$ with

$$\begin{aligned} \mathcal{W}_{[0, T]} \triangleq \left\{ w: [0, T] \times \Omega \rightarrow W \mid w \text{ is an } \{\mathcal{F}_t\}_{t \geq 0} \text{-adapted} \right. \\ \left. \text{and } \mathbb{E} \int_0^T |w_t|^2 dt < \infty \right\}. \end{aligned} \quad (1.4)$$

Furthermore, we consider the following two cost functionals that provide information about the accumulated risk-costs on the time interval $[0, T]$ w.r.t. the strategies of the *leader* and that of the *follower*, i.e.,

$$\text{leader's accu. risk-cost: } \xi_{0, T}^1(v, w) = \int_0^T c_1(t, X_t, v_t) dt + \Psi_1(X_T) \quad (1.5)$$

and

$$\text{follower's accu. risk-cost: } \xi_{0, T}^2(v, w) = \int_0^T c_2(t, X_t, w_t) dt + \Psi_2(X_T), \quad (1.6)$$

where the cost-rate functionals $c_1: [0, T] \times \mathbb{R}^d \times V \rightarrow \mathbb{R}$ and $c_2: [0, T] \times \mathbb{R}^d \times W \rightarrow \mathbb{R}$ are measurable functions; and the final-stage risk-costs $\Psi_i: \mathbb{R}^d \rightarrow \mathbb{R}$, with $i = 1, 2$, (associated with risk-averse satisfaction levels) are also assumed measurable functions.

Here, we remark that the corresponding solution X_t in (1.1) (i.e., $X_t = X_t^{0, x; u}$ with $u. \equiv (v., w.) \in \mathcal{V}_{[0, T]} \otimes \mathcal{W}_{[0, T]}$) depends on the admissible risk-averse strategies of the *leader* and that of the *follower*; and it also depends on the initial condition $X_0 = x$. As a result of this, for any time-interval $[t, T]$, with $t \in [0, T]$, the accumulated risk-costs $\xi_{t, T}^1$ and $\xi_{t, T}^2$ (cf.

equations (1.5) and (1.6)) depend on the risk-averse strategies $(v, w) \in \mathcal{V}_{[t,T]} \otimes \mathcal{W}_{[t,T]}$.¹ Moreover, we also assume that f , σ , c_i and Ψ_i , $i = 1, 2$, satisfy the following linear growth conditions

$$\begin{aligned} & |f(t, x, (v, w))| + |\sigma(t, x, (v, w))| + |c_1(t, x, v)| + |\Psi_1(x)| \\ & \leq K(1 + |x| + |v| + |w|) \end{aligned} \quad (1.7)$$

and

$$\begin{aligned} & |f(t, x, (v, w))| + |\sigma(t, x, (v, w))| + |c_2(t, x, w)| + |\Psi_2(x)| \\ & \leq K(1 + |x| + |v| + |w|), \end{aligned} \quad (1.8)$$

for all $(t, x, (v, w)) \in [0, T] \times \mathbb{R}^d \times (V \times W)$ and for some constant $K > 0$.

Next, we introduce the following measurable spaces that will be useful later in the paper.

- $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ is the set of \mathbb{R}^d -valued \mathcal{F}_t -measurable random variables ξ such that $\|\xi\|^2 = \mathbb{E}\{|\xi|^2\} < \infty$;
- $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ is the set of \mathbb{R} -valued \mathcal{F}_t -measurable random variables ξ such that $\|\xi\| = \text{ess inf}|\xi| < \infty$;
- $\mathcal{S}^2(t, T; \mathbb{R}^d)$ is the set of \mathbb{R}^d -valued adapted processes $(\varphi_s)_{t \leq s \leq T}$ on $\Omega \times [t, T]$ such that $\|\varphi\|_{[t,T]}^2 = \mathbb{E}\{\sup_{t \leq s \leq T} |\varphi_s|^2\} < \infty$;
- $\mathcal{H}^2(t, T; \mathbb{R}^d)$ is the set of \mathbb{R}^d -valued progressively measurable processes $(\varphi_s)_{t \leq s \leq T}$ such that $\|\varphi\|_{[t,T]}^2 = \mathbb{E}\{\int_t^T |\varphi_s|^2 ds\} < \infty$.

On the same probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, we consider the following one-dimensional backward stochastic differential equation (BSDE)

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dB_t, \quad Y_T = \xi, \quad (1.9)$$

where the terminal value $Y_T = \xi$ belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ and the generator function $g: \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, with property that $(g(t, y, z))_{0 \leq t \leq T}$ is progressively measurable for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. We also assume that g satisfies the following assumption.

ASSUMPTION 1.1.

- (A1) g is Lipschitz in (y, z) , i.e., there exists a constant $K > 0$ such that, \mathbb{P} -a.s., for any $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq K(|y_1 - y_2| + \|z_1 - z_2\|).$$

- (A2) $g(t, 0, 0) \in \mathcal{H}^2(t, T; \mathbb{R})$.

- (A3) \mathbb{P} -a.s., for all $t \in [0, T]$ and $y \in \mathbb{R}$, $g(t, y, 0) = 0$.

¹For any $t \in [0, T]$, $\mathcal{V}_{[t,T]}$ and $\mathcal{W}_{[t,T]}$ respectively denote the sets of V - and W -valued $\{\mathcal{F}_s^t\}_{s \geq t}$ -adapted processes (see Definition 2.1).

Then, we state the following lemma, which is used to establish the existence of a unique adapted solution (e.g., see [18] for additional discussions).

LEMMA 1.2. *Suppose that Assumption 1.1 holds. Then, for any $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, the BSDE in (1.9), with terminal condition $Y_T = \xi$, i.e.,*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T \quad (1.10)$$

has a unique adapted solution

$$(Y_t^{T,g,\xi}, Z_t^{T,g,\xi})_{0 \leq t \leq T} \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d). \quad (1.11)$$

Moreover, we recall the following comparison result that will be useful later (e.g., see [19]).

THEOREM 1.3 (Comparison Theorem). *Given two generators g_1 and g_2 satisfying Assumption 1.1 and two terminal conditions $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Let (Y_t^1, Z_t^1) and (Y_t^2, Z_t^2) be the solution pairs corresponding to (ξ_1, g_1) and (ξ_2, g_2) , respectively. Then, we have*

- (i) *Monotonicity: If $\xi_1 > \xi_2$ and $g_1 > g_2$, \mathbb{P} -a.s., then $Y_t^1 > Y_t^2$, \mathbb{P} -a.s., for all $t \in [0, T]$;*
- (ii) *Strictly Monotonicity: In addition to (i) above, if we assume that $\mathbb{P}(\xi_1 > \xi_2) > 0$, then $\mathbb{P}(Y_t^1 > Y_t^2) > 0$, for all $t \in [0, T]$.*

In the following, we give a definition for a dynamic risk measure that is associated with the generator of BSDE in (1.9).

DEFINITION 1.4. *For any $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, let $(Y_t^{T,g,\xi}, Z_t^{T,g,\xi})_{0 \leq t \leq T} \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d)$ be the unique solution to the BSDE in (1.9) with terminal condition $Y_T = \xi$. Then, we define the dynamic risk measure $\rho_{t,T}^g$ of ξ as*

$$\rho_{t,T}^g[\xi] \triangleq Y_t^{T,g,\xi}. \quad (1.12)$$

REMARK 1.5. *Note that such a risk measure is widely used for evaluating the risk of stochastic processes or uncertain outcomes, and assists with stipulating minimum interventions required by financial institutions for risk management (e.g., see [2], [22], [8], [10], [6] or [4] for related discussions). In Section 2, we use a family of dynamic risk measures associated with the leader's and follower's cost functionals and risk-averse satisfactions (cf. Property 1.6 below); and we provide a hierarchical framework for the risk-averse control problem.*

Note that if the generator g satisfies Assumption 1.1, then a family of time-consistent dynamic convex risk measures $\{\rho_{t,T}^g\}_{t \in [0, T]}$ has the following properties (see [22] for additional discussions).

PROPERTY 1.6.

- (P1) *Normalization: $\rho_{t,T}^g[0] = 0$ for $t \in [0, T]$;*
- (P2) *Monotonicity: For $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ such that $\xi_1 > \xi_2$ \mathbb{P} -a.s., then*

$$\rho_{t,T}^g[\xi_1] > \rho_{t,T}^g[\xi_2], \quad \mathbb{P}\text{-a.s.};$$

(P3) *Translation Invariance:* For all $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ and $\nu \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$

$$\rho_{t,T}^g[\xi + \nu] = \rho_{t,T}^g[\xi] + \nu;$$

(P4) *Convexity:* If g is convex function for every fixed $(t, \omega) \in [0, T] \times \Omega$, then for all $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ and for all $\lambda \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ such that $0 \leq \lambda \leq 1$

$$\rho_{t,T}^g[\lambda \xi_1 + (1 - \lambda) \xi_2] \leq \lambda \rho_{t,T}^g[\xi_1] + (1 - \lambda) \rho_{t,T}^g[\xi_2];$$

(P5) *Positive Homogeneity:* For all $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ and for all $\lambda \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ such that $\lambda > 0$

$$\rho_{t,T}^g[\lambda \xi] = \lambda \rho_{t,T}^g[\xi].$$

REMARK 1.7. Note that, since the seminal work of Artzner et al. [2], there have been studies on axiomatic dynamic risk measures, coherency and consistency in the literature (e.g., see [6], [22], [24], [10] or [4]). Particularly relevant for us is the time-consistent dynamic convex risk measures, based on the backward-semigroup operators associated with generators of BSDEs in (2.8) and (2.9), that satisfy the above properties (P1)–(P5).

Here it is worth mentioning that some interesting studies on the dynamic risk measures, based on the backward-semigroup operators, have been reported in the literature (e.g. see [22], [4] and [24] for establishing connection between the risk measures and the generator of BSDE; and see also [26] for characterizing the generator of BSDE according to different risk measures). Recently, the author in [25] has provided interesting results on the risk-averse control problem for Markov decision processes in discrete-time setting. Note that the rationale behind our framework follows in some sense the settings of these papers. However, to our knowledge, the problem of optimal risk-averse control for diffusion processes has not been addressed in the context of hierarchical argument, and it is important because it provides a mathematical framework that shows how a hierarchical framework can be systematically used to obtain optimal risk-averse strategies for such controlled-diffusion processes.²

The remainder of this paper is organized as follows. In Section 2, using the basic remarks made in Section 1, we state the hierarchical risk-averse control problem for the diffusion process. In Section 3, we present our main results – where we introduce a framework under which the *follower* is required to *respond optimally* to the risk-averse strategy of the *leader* so as to achieve the overall risk-averseness. Moreover, we establish the existence of optimal risk-averse solutions, in the sense of viscosity solutions, to the associated risk-averse dynamic programming equations. Finally, Section 4 provides further remarks.

2. The hierarchical risk-averse control problem formulation. In this section, we formulate the problem of risk-averse control, in which there are two decision-making groups, namely, the *leader* and *follower*, with different cost functionals and risk-averse satisfactions. In order to make our formulation more precise, for any $(t, x) \in [0, T] \times \mathbb{R}^d$, we consider the following forward SDE with an initial condition $X_t^{t,x;u} = x$

$$dX_s^{t,x;u} = f(s, X_s^{t,x;u}, (v_s, w_s))ds + \sigma(s, X_s^{t,x;u}, (v_s, w_s))dB_s, \quad t \leq s \leq T, \quad (2.1)$$

²In this paper, our intent is to provide a theoretical framework, rather than considering a specific numerical problem or application.

where $u. = (v., w.)$ are (V, W) -valued measurable control processes (cf. equations (1.3) and (1.4)). Moreover, we introduce the following two risk-value functions w.r.t. the strategies of the *leader* and that of the *follower*, i.e.,

$$\begin{aligned} \text{leader: } V_1^v(t, x) &= \rho_{t,T}^{g_1}[\xi_{t,T}^1(v, w)], \\ &\text{such that } w \in \left\{ \tilde{w}. \in \mathcal{W}_{[t,T]} \mid \text{given } \hat{v}. \in \mathcal{V}_{[t,T]} \text{ and} \right. \\ &\quad \left. \rho_{t,T}^{g_2}[\xi_{t,T}^2(\hat{v}, \tilde{w})] < \rho_{t,T}^{g_2}[\xi_{t,T}^2(\hat{v}, w)], \forall w. \in \mathcal{W}_{[t,T]} \right\}, \end{aligned} \quad (2.2)$$

where

$$\xi_{t,T}^1(v, w) = \int_t^T c_1(s, X_s^{t,x;u}, v_s) ds + \Psi_1(X_T^{t,x;u}) \quad (2.3)$$

and similarly

$$\text{follower: } V_2^w(t, x) = \rho_{t,T}^{g_2}[\xi_{t,T}^2(v, w)], \quad (2.4)$$

where

$$\xi_{t,T}^2(v, w) = \int_t^T c_2(s, X_s^{t,x;u}, w_s) ds + \Psi_2(X_T^{t,x;u}). \quad (2.5)$$

Note that we can express the above risk-value functions in (2.2) and (2.4) as follow

$$\begin{aligned} V_1^v(t, x) &= \xi_{t,T}^1(v, w) + \int_t^T g_1(s, Y_s^{1;t,x;u}, Z_s^{1;t,x;u}) ds - \int_t^T Z_s^{1;t,x;u} dB_s, \\ &= \Psi_1(X_T^{t,x;u}) + \int_t^T \left\{ c_1(s, X_s^{t,x;u}, v_s) + g_1(s, Y_s^{1;t,x;u}, Z_s^{1;t,x;u}) \right\} ds \\ &\quad - \int_t^T Z_s^{1;t,x;u} dB_s \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} V_2^w(t, x) &= \xi_{t,T}^2(v, w) + \int_t^T g_2(s, Y_s^{2;t,x;u}, Z_s^{2;t,x;u}) ds - \int_t^T Z_s^{2;t,x;u} dB_s, \\ &= \Psi_2(X_T^{t,x;u}) + \int_t^T \left\{ c_2(s, X_s^{t,x;u}, w_s) + g_2(s, Y_s^{2;t,x;u}, Z_s^{2;t,x;u}) \right\} ds \\ &\quad - \int_t^T Z_s^{2;t,x;u} dB_s, \end{aligned} \quad (2.7)$$

where the generators g_1 and g_2 are assumed to satisfy Assumption 1.1. Further, noting the conditions in (1.7) and (1.8), then $(Y_s^{1;t,x;u}, Z_s^{1;t,x;u})_{t \leq s \leq T}$ and $(Y_s^{2;t,x;u}, Z_s^{2;t,x;u})_{t \leq s \leq T}$ are adapted solutions on $[t, T] \times \Omega$ and belong to $\mathcal{S}^2(t, T; \mathbb{R}) \times \mathcal{H}^2(t, T; \mathbb{R}^d)$. Equivalently, we can also rewrite (2.6) and (2.7) as a family of coupled BSDEs on the probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$, i.e.,

$$\begin{aligned} -dY_s^{1;t,x;u} &= \left\{ c_1(s, X_s^{t,x;u}, v_s) + g_1(s, Y_s^{1;t,x;u}, Z_s^{1;t,x;u}) \right\} ds - Z_s^{1;t,x;u} dB_s, \\ &\quad s \in [t, T], \quad Y_T^1 = \Psi_1(X_T^{t,x;u}) \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} -dY_s^{2;t,x;u} &= \left\{ c_2(s, X_s^{t,x;u}, w_s) + g_2(s, Y_s^{2;t,x;u}, Z_s^{2;t,x;u}) \right\} ds - Z_s^{2;t,x;u} dB_s, \\ s &\in [t, T], \quad Y_T^2 = \Psi_2(X_T^{t,x;u}). \end{aligned} \quad (2.9)$$

In what follows, we introduce a hierarchical framework that requires a certain level of risk-averse satisfaction be achieved for the *leader* as a priority over that of the *follower's* risk-averseness. For example, suppose that the risk-averse strategy of the *leader* $\hat{v} \in \mathcal{V}_{[t,T]}$ is given. Then, the problem of finding an optimal risk-averse strategy for the *follower*, i.e., $\hat{w} \in \mathcal{W}_{[t,T]}$, which minimizes the accumulated risk-cost under w is then reduced to finding an optimal risk-averse solution for

$$\inf_{w \in \mathcal{W}_{[t,T]}} J_2[(\hat{v}, w)], \quad (2.10)$$

where

$$J_2[(\hat{v}, w)] = \rho_{t,T}^{g_2}[\xi_{t,T}^2(\hat{v}, w)]. \quad (2.11)$$

Note that, for a given $\hat{v} \in \mathcal{V}_{[t,T]}$, if the forward-backward stochastic differential equations (FBSDEs) in (2.1), (2.8) and (2.9) admit weak solutions, then we have

$$\begin{aligned} \hat{w} \in S(\hat{v}) &\subset \left\{ \tilde{w} \in \mathcal{W}_{[t,T]} \mid \text{given } \hat{v} \in \mathcal{V}_{[t,T]} \text{ and} \right. \\ &\quad \left. \rho_{t,T}^{g_2}[\xi_{t,T}^2(\hat{v}, \tilde{w})] < \rho_{t,T}^{g_2}[\xi_{t,T}^2(\hat{v}, w)], \forall w \in \mathcal{W}_{[t,T]} \right\} \end{aligned} \quad (2.12)$$

for some (nonanticipating) measurable mapping $S: \mathcal{V}_{[t,T]} \rightrightarrows \mathcal{W}_{[t,T]}$. Further, if we substitute $u = (\hat{v}, S(\hat{v}))$ into (2.1), then the corresponding solution $X_s^{t,x;u}$ depends uniformly on $\hat{v} \in \mathcal{V}_{[t,T]}$ for $s \in [t, T]$.³ Moreover, the risk-averse control problem (which minimizes the accumulated risk-cost under v w.r.t the *leader*) is then reduced to finding an optimal risk-averse solution for

$$\inf_{v \in \mathcal{V}_{[t,T]}} J_1[(v, S(v))], \quad (2.13)$$

where

$$J_1[(v, S(v))] = \rho_{t,T}^{g_1}[\xi_{t,T}^1(v, S(v))]. \quad (2.14)$$

Next, we introduce the definition of admissible hierarchical risk-averse control system $\Sigma_{[t,T]}$ which provides a logical construct for our main results (e.g., see [16]).

DEFINITION 2.1. *For a given finite-time horizon $T > 0$, we call $\Sigma_{[t,T]}$ an admissible hierarchical risk-averse control system, if it satisfies the following conditions:*

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a complete probability space;
- $\{B_s\}_{s \geq t}$ is a d -dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ over $[t, T]$ and $\mathcal{F}^t \triangleq \{\mathcal{F}_s^t\}_{s \in [t, T]}$, where $\mathcal{F}_s^t = \sigma\{B_s; t \leq s \leq T\}$ is augmented by all \mathbb{P} -null sets in \mathcal{F} ;

³In this paper, for the sake of simplicity, we use the same notation to represent both the set in (2.12) and an element of that set which is uniquely selectable.

- $v : \Omega \times [s, T] \rightarrow V$ and $w : \Omega \times [s, T] \rightarrow W$ are $\{\mathcal{F}_s^t\}_{s \geq t}$ -adapted processes on $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{E} \int_s^T |v_\tau|^2 d\tau < \infty \quad \text{and} \quad \mathbb{E} \int_s^T |w_\tau|^2 d\tau < \infty, \quad s \in [t, T];$$

- There exists at least one measurable mapping $S : \mathcal{V}_{[t, T]} \rightrightarrows \mathcal{W}_{[t, T]}$ with $w. \in S(v.)$ whenever $v. \in \mathcal{V}_{[t, T]}$;
- For any $x \in \mathbb{R}^d$, the FBSDEs in (2.1), (2.8) and (2.9) admit a unique solution set $\{X^{s, x; u}, (Y^{1; s, x; u}, Z^{1; s, x; u}), (Y^{2; s, x; u}, Z^{2; s, x; u})\}$ on $(\Omega, \mathcal{F}, \mathcal{F}^t, \mathbb{P})$ with $u = (v, S(v))$.

Then, for the admissible hierarchical risk-averse control system $\Sigma_{[0, T]}$, we can state the problem of risk-averse control as follow.

Problem (P). Find a pair of risk-averse strategies $(v^*, w^*) \in \mathcal{V}_{[0, T]} \otimes \mathcal{W}_{[0, T]}$ w.r.t. the *leader* and that of the *follower* such that

$$v^* \in \left\{ \arg \inf J_1[(v, w)] \mid w. \in S(v.) \text{ \& } (v., S(v.)) \text{ restricted to } \Sigma_{[0, T]} \right\} \subset \mathcal{V}_{[0, T]} \quad (2.15)$$

and

$$w^* \in \left\{ \arg \inf J_2[(v, w)] \mid w^* \in S(v^*) \text{ \& } (v^*, S(v^*)) \text{ restricted to } \Sigma_{[0, T]} \right\} \subset \mathcal{W}_{[0, T]} \quad (2.16)$$

where S is a unique measurable mapping that maps $\mathcal{V}_{[0, T]}$ onto $\mathcal{W}_{[0, T]}$ and, furthermore, the accumulated risk-costs J_1 and J_2 over the time-interval $[0, T]$ are given

$$J_1[(v, w)] = \int_0^T c_1(s, X_s^{0, x; u}, v_s) ds + \Psi_1(X_T^{0, x; u}) \quad (2.17)$$

and

$$J_2[(v, w)] = \int_0^T c_2(s, X_s^{0, x; u}, w_s) ds + \Psi_2(X_T^{0, x; u}), \quad (2.18)$$

where $X_0^{0, x; u} = x$ and $u = (v, w)$.⁴

In the following section, we establish the existence of optimal risk-averse solutions, in the sense of viscosity solutions, for the optimization problems in (2.15) and (2.16) with restriction

⁴We remark that, for the above optimization problems in (2.15) and (2.16), if there exists a nonanticipating measurable mapping $S : \mathcal{V}_{[0, T]} \rightrightarrows \mathcal{W}_{[0, T]}$ such that, for any given $\hat{v}. \in \mathcal{V}_{[0, T]}$ (i.e., a risk-averse strategy for the *leader*),

$$\rho_{0, T}^{g_2}[\xi_{0, T}^2(\hat{v}, S(\hat{v}))] < \rho_{0, T}^{g_2}[\xi_{0, T}^2(\hat{v}, w)], \quad \forall w. \in \mathcal{W}_{[0, T]} \quad (2.19)$$

and if there exists a $v^* \in \mathcal{V}_{[0, T]}$ such that

$$\rho_{0, T}^{g_1}[\xi_{0, T}^1(v^*, S(v^*))] < \rho_{0, T}^{g_1}[\xi_{0, T}^1(v, S(v))], \quad \forall v. \in \mathcal{V}_{[0, T]}. \quad (2.20)$$

Then, the pair $(v^*, w^*) \in \mathcal{V}_{[0, T]} \otimes \mathcal{W}_{[0, T]}$, with $w^* \in S(v^*)$, is an optimal risk-averse strategy. Further, note that the risk-averse strategy for the *leader* always guarantees him an accumulated risk-cost that can not be exceeded, no matter what the *follower's rational-response* (see also Section 4).

to $\Sigma_{[0,T]}$. Note that, for a given $v. \in \mathcal{V}_{[0,T]}$, the optimization problem in (2.16) has a unique solution on $\mathcal{W}_{[0,T]}$ (see Proposition 3.8). Moreover, as we will see later on (particularly in Proposition 3.10), the problem in (2.15) makes sense if the *follower* is involved not only in minimizing his own accumulated risk-cost (in response to the risk-averse strategy of the *leader*) but also in minimizing that of the *leader*.

3. Main results. In this section, we present our main results, where we introduce a framework under which the *follower* is required to respond optimally to the risk-averse strategy of the *leader* so as to achieve an overall risk-averseness. Moreover, such a framework allows us to establish the existence of optimal risk-averse solutions, in the sense of viscosity solutions, to the associated risk-averse dynamic programming equations.

We now state the following propositions that will be useful for proving our main results (i.e., in Subsections 3.1 and 3.2).

PROPOSITION 3.1. *Suppose Assumption 1.1 together with (1.7) and (1.8) hold. Then, for any $(t, x) \in [0, T] \times \mathbb{R}^d$ and for every $(v., w.) \in \mathcal{V}_{[t,T]} \otimes \mathcal{W}_{[t,T]}$, the FBSDEs in (2.1), (2.8) and (2.9) admit unique adapted solutions*

$$\left. \begin{aligned} X^{t,x;u} &\in \mathcal{S}^2(t, T; \mathbb{R}) \\ (Y^{1;t,x;u}, Z^{1;t,x;u}) &\in \mathcal{S}^2(t, T; \mathbb{R}) \times \mathcal{H}^2(t, T; \mathbb{R}^d) \\ (Y^{2;t,x;u}, Z^{2;t,x;u}) &\in \mathcal{S}^2(t, T; \mathbb{R}) \times \mathcal{H}^2(t, T; \mathbb{R}^d) \end{aligned} \right\} \quad (3.1)$$

Furthermore, the risk-values w.r.t. the leader and follower, i.e., $V_1^v(t, x)$ and $V_2^w(t, x)$, are deterministic.

Proof. Notice that f and σ are bounded and Lipschitz continuous w.r.t. $(t, x) \in [0, T] \times \mathbb{R}^d$ and uniformly for $(v, w) \in V \times W$. Then, for any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $u. = (v., w.)$ are progressively measurable processes, there always exists a unique path-wise solution $X^{t,x;u} \in \mathcal{S}^2(t, T; \mathbb{R})$ for the forward SDE in (2.1). On the other hand, consider the following BSDEs

$$-d\hat{Y}_s^{1;t,x;u} = g_1(s, Z_s^{1;t,x;u})ds - Z_s^{1;t,x;u}dB_s, \quad (3.2)$$

where

$$\hat{Y}_T^{1;t,x;u} = \int_t^T c_1(\tau, X_\tau^{t,x;u}, v_\tau)d\tau + \Psi_1(X_T^{t,x;u})$$

and

$$-d\hat{Y}_s^{2;t,x;u} = g_2(s, Z_s^{2;t,x;u})ds - Z_s^{2;t,x;u}dB_s, \quad (3.3)$$

where

$$\hat{Y}_T^{2;t,x;u} = \int_t^T c_2(\tau, X_\tau^{t,x;u}, w_\tau)d\tau + \Psi_2(X_T^{t,x;u}).$$

From Lemma 1.2, the equations in (3.2) and (3.3) admit unique solutions $(\hat{Y}^{1;t,x;u}, Z^{1;t,x;u})$ and $(\hat{Y}^{2;t,x;u}, Z^{2;t,x;u})$ in $\mathcal{S}^2(t, T; \mathbb{R}) \times \mathcal{H}^2(t, T; \mathbb{R}^d)$. Furthermore, if we introduce the following

$$Y_s^{1;t,x;u} = \hat{Y}_s^{1;t,x;u} - \int_t^s c_1(\tau, X_\tau^{t,x;u}, v_\tau)d\tau, \quad s \in [t, T]$$

and

$$Y_s^{2;t,x;u} = \hat{Y}_s^{2;t,x;u} - \int_t^s c_2(\tau, X_\tau^{t,x;u}, w_\tau) d\tau, \quad s \in [t, T].$$

Then, the forward SDE in (2.8) and (2.9) hold, with $(Y^{1;t,x;u}, Z^{1;t,x;u})$ and $(Y^{2;t,x;u}, Z^{2;t,x;u})$, respectively. Moreover, we also observe that $Y_t^{1;t,x;u}$ and $Y_t^{2;t,x;u}$ are deterministic. This completes the proof of Proposition 3.1. \square

PROPOSITION 3.2. *Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and $(v, w) \in \mathcal{V}_{[t,T]} \otimes \mathcal{W}_{[t,T]}$ be restricted to $\Sigma_{[t,T]}$ (cf. Definition 2.1). Then, for any $r \in [t, T]$ and \mathbb{R}^d -valued \mathcal{F}_r^t -measurable random variable η , we have*

$$V_1^v(r, \eta) = \rho_{r,T}^{g_1} \left[\int_r^T c_1(s, X_s^{r,\eta;u}, v_s) ds + \Psi_1(X_T^{r,\eta;u}) \right], \quad P\text{-a.s.} \quad (3.4)$$

and

$$V_2^w(r, \eta) = \rho_{r,T}^{g_2} \left[\int_r^T c_2(s, X_s^{r,\eta;u}, w_s) ds + \Psi_2(X_T^{r,\eta;u}) \right], \quad P\text{-a.s.} \quad (3.5)$$

Proof. For any $r \in [t, T]$, with $t \in [0, T]$, we consider the following probability space $(\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_r^t), \{\mathcal{F}^t\})$ and notice that η is deterministic under this probability space. Then, for any $s \geq r$, there exist progressively measurable processes ψ_1 and ψ_2 such that

$$(v_s(\Omega), w_s(\Omega)) = (\psi_1(\Omega, B_{\cdot \wedge s}(\Omega)), \psi_2(\Omega, B_{\cdot \wedge s}(\Omega))), \quad (3.6)$$

$$= (\psi_1(s, \bar{B}_{\cdot \wedge s}(\Omega) + B_r(\Omega)), \psi_2(s, \bar{B}_{\cdot \wedge s}(\Omega) + B_r(\Omega))), \quad (3.7)$$

where $\bar{B}_s = B_s - B_r$ is a standard d -dimensional brownian motion. Note that the pairs (v, w) are \mathcal{F}_r^t -adapted processes, then we have the following restriction w.r.t. $\Sigma_{[t,T]}$

$$(\Omega, \mathcal{F}, \{\mathcal{F}^t\}, \mathbb{P}(\cdot|\mathcal{F}_r^t)(\omega'), B, (v, w)) \in \Sigma_{[t,T]}, \quad (3.8)$$

where $\omega' \in \Omega'$ such that $\Omega' \in \mathcal{F}$, with $\mathbb{P}(\Omega') = 1$. Furthermore, noting Lemma 1.2, if we work under the probability space $(\Omega', \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_r^t))$, then both statements in (3.4) and (3.5) hold \mathbb{P} -almost surely. This completes the proof of Proposition 3.2. \square

In what follows, we restrict our discussion when the generators g_1 and g_2 depend only on $(t, z) \in [0, T] \times \mathbb{R}^d$. Moreover, for $(v, w) \in V \times W$ and any $\phi(x) \in C_0^\infty(\mathbb{R}^d)$, we introduce a family of second-order linear operators, associated with (1.1), as follow

$$\mathcal{L}_t^{(v,w)} \phi(x) = \frac{1}{2} \text{tr} \left\{ a(t, x, u) D_x^2 \phi(x) \right\} + f(t, x, u) D_x \phi(x), \quad t \in [0, T], \quad (3.9)$$

where $a(t, x, u) = \sigma(t, x, u) \sigma^T(t, x, u)$, D_x and D_x^2 , (with $D_x^2 = (\partial^2 / \partial x_i \partial x_j)$) are the gradient and the Hessian (w.r.t. the variable x), respectively. Further, on the space $C_b^{1,2}([t, T] \times \mathbb{R}^d)$, for any $(t, x) \in [0, T] \times \mathbb{R}^d$, we consider the following coupled Hamilton-Jacobi-Bellman (HJB) partial differential equations

$$\left. \begin{aligned} \frac{\partial \varphi_1(t, x)}{\partial t} + \inf_{v \in V} \left\{ c_1(t, x, v) + \mathcal{L}_t^{(v,w)} \varphi_1(t, x) \right. \\ \left. + g_1(t, D_x \varphi_1(t, x) \cdot \sigma(t, x, (v, w))) \right\} = 0 \\ \text{where } w \text{ is assumed to be fixed} \end{aligned} \right\} \quad (3.10)$$

and

$$\left. \begin{aligned} \frac{\partial \varphi_2(t, x)}{\partial t} + \inf_{w \in W} \left\{ c_2(t, x, w) + \mathcal{L}_t^{(v, w)} \varphi_2(t, x) \right. \\ \left. + g_2(t, D_x \varphi_2(t, x) \cdot \sigma(t, x, (v, w))) \right\} = 0 \\ \text{where } v \text{ is assumed to be given} \end{aligned} \right\} \quad (3.11)$$

with, respectively, the following boundary conditions

$$\varphi_1(T, x) = \Psi_1(T, x) \quad \text{and} \quad \varphi_2(T, x) = \Psi_2(T, x), \quad x \in \mathbb{R}^d. \quad (3.12)$$

REMARK 3.3. *Here, we remark that the above equations in (3.10) and (3.11) together with (3.12), are associated with the risk-averse control problem w.r.t. the leader and follower, restricted to $\Sigma_{[t, T]}$ (cf. Definition 2.1), with cost functionals in (2.17) and (2.18). Moreover, they represent generalized HJB equations with additional terms g_1 and g_2 , respectively. Note that the problem of FBSDEs (cf. equations (2.1), (2.8) and (2.9)) and the solvability of the HJB partial differential equations have been well studied in literature (e.g., see [1], [14], [16], [17], [19], [20] and [21]).*

Next, we recall the definitions of viscosity solutions for (3.10) and (3.11) together with (3.12) (e.g., see [5], [9] or [15] for additional discussions on the notion of viscosity solutions).

DEFINITION 3.4. *The functions $\varphi_i: [0, T] \times \mathbb{R}^d$, with $i = 1, 2$, are viscosity solutions for (3.10) and (3.11) together with the boundary conditions in (3.12), if the following conditions hold*

(i) *for every $\psi_i \in C_b^{1,2}([0, T], \times \mathbb{R}^d)$, with $i = 1, 2$, such that $\psi_i \geq \varphi_i$ on $[0, T] \times \mathbb{R}^d$,*

$$\sup_{(t, x)} \{ \varphi_i(t, x) - \psi_i(t, x) \} = 0, \quad (3.13)$$

and for $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ such that $\psi_i(t_0, x_0) = \varphi_i(t_0, x_0)$ (i.e., a local maximum at (t_0, x_0)), then we have

$$\begin{aligned} \frac{\partial \psi_1(t_0, x_0)}{\partial t} + \inf_{v \in V} \left\{ c_1(t_0, x_0, v) + \mathcal{L}_t^{(v, w)} \psi_1(t_0, x_0) \right. \\ \left. + g_1(t_0, D_x \psi_1(t_0, x_0) \cdot \sigma(t_0, x_0, (v, w))) \right\} \geq 0 \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} \frac{\partial \psi_2(t_0, x_0)}{\partial t} + \inf_{w \in W} \left\{ c_2(t_0, x_0, w) + \mathcal{L}_t^{(v, w)} \psi_2(t_0, x_0) \right. \\ \left. + g_2(t_0, D_x \psi_2(t_0, x_0) \cdot \sigma(t_0, x_0, (v, w))) \right\} \geq 0 \end{aligned} \quad (3.15)$$

(ii) *for every $\psi_i \in C_b^{1,2}([0, T], \times \mathbb{R}^d)$, with $i = 1, 2$, such that $\psi_i \leq \varphi_i$ on $[0, T] \times \mathbb{R}^d$,*

$$\inf_{(t, x)} \{ \varphi_i(t, x) - \psi_i(t, x) \} = 0, \quad (3.16)$$

and for $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ such that $\psi_i(t_0, x_0) = \varphi_i(t_0, x_0)$ (i.e., a local minimum at (t_0, x_0)), then we have

$$\begin{aligned} \frac{\partial \psi_1(t_0, x_0)}{\partial t} + \inf_{v \in V} \left\{ c_1(t_0, x_0, v) + \mathcal{L}_t^{(v, w)} \psi_1(t_0, x_0) \right. \\ \left. + g_1(t_0, D_x \psi_1(t_0, x_0) \cdot \sigma(t_0, x_0, (v, w))) \right\} \leq 0 \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} \frac{\partial \psi_2(t_0, x_0)}{\partial t} + \inf_{w \in W} \left\{ c_2(t_0, x_0, w) + \mathcal{L}_t^{(v, w)} \psi_2(t_0, x_0) \right. \\ \left. + g_2(t_0, D_x \psi_2(t_0, x_0) \cdot \sigma(t_0, x_0, (v, w))) \right\} \leq 0. \end{aligned} \quad (3.18)$$

3.1. On the risk-averse optimality condition for the follower. Suppose that, for a given *leader's* risk-averse strategy $\hat{v} \in \mathcal{V}_{[t, T]}$, the risk-averse strategy for the *follower* is an optimal solution to (2.7) (cf. equations (2.3) and (2.16)). Then, with restriction to $\Sigma_{[t, T]}$, such a solution is characterized by the following propositions (i.e., Propositions 3.8, 3.9 and 3.10).

PROPOSITION 3.5. *Suppose that Assumption 1.1 together with (1.7) and (1.8) hold. Let $\hat{v} \in \mathcal{V}_{[t, T]}$ be given, then the risk-value function w.r.t. the follower is given by*

$$V_2^w(t, x) = \inf_{w \in \mathcal{W}_{[t, r]}|_{\Sigma_{[t, T]}}} \rho_{t, r}^{g_2} \left[\int_t^r c_2(s, X_s^{t, x; u}, w_s) ds + V_2^w(r, X_r^{t, x; u}) \right] \quad (3.19)$$

for any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $r \in [t, T]$, with $u = (\hat{v}, w)$.

Proof. Notice that $\hat{v} \in \mathcal{V}_{[t, T]}$ is given. Then, for any $\epsilon > 0$, there exists $\tilde{w} \in \mathcal{W}_{[t, T]}$ such that $V_2^w(t, x) + \epsilon \geq V_2^{\tilde{w}}(t, x)$. Further, if we applying the properties of time-consistency and translation to $V_2^{\tilde{w}}(t, x)$, then we have

$$\begin{aligned} V_2^w(t, x) + \epsilon &\geq V_2^{\tilde{w}}(t, x) \\ &= \rho_{t, r}^{g_2} \left[\rho_{r, T}^{g_2} \left[\int_t^T c_2(s, X_s^{t, x; \tilde{u}}, \tilde{w}_s) ds + \Psi_2(X_T^{t, x; \tilde{u}}) \right] \right] \\ &= \rho_{t, r}^{g_2} \left[\int_t^r c_2(s, X_s^{t, x; \tilde{u}}, \tilde{w}_s) ds + \rho_{r, T}^{g_2} \left[\int_r^T c_2(s, X_s^{t, x; \tilde{u}}, \tilde{w}_s) ds + \Psi_2(X_T^{t, x; \tilde{u}}) \right] \right], \end{aligned} \quad (3.20)$$

where $\tilde{u} = (\hat{v}, \tilde{w})$ is restricted to $\Sigma_{[t, T]}$. Moreover, if we apply Proposition 3.2, then we have

$$\begin{aligned} V_2^w(t, x) + \epsilon &\geq \rho_{t, r}^{g_2} \left[\int_t^r c_2(s, X_s^{t, x; \tilde{u}}, \tilde{w}_s) ds + V_2^{\tilde{w}}(r, X_r^{t, x; \tilde{u}}) \right] \\ &\geq \rho_{t, r}^{g_2} \left[\int_t^r c_2(s, X_s^{t, x; \tilde{u}}, \tilde{w}_s) ds + V_2^w(r, X_r^{t, x; \tilde{u}}) \right] \\ &\geq \inf_{w \in \mathcal{W}_{[t, r]}|_{\Sigma_{[t, T]}}} \rho_{t, r}^{g_2} \left[\int_t^r c_2(s, X_s^{t, x; \tilde{u}}, \tilde{w}_s) ds + V_2^w(r, X_r^{t, x; u}) \right]. \end{aligned} \quad (3.21)$$

Since ϵ is arbitrary, we obtain (3.19). On the other hand, to show the reverse inequality “ \leq ”, let \tilde{w} . (which is restricted to $\Sigma_{[t,T]}$) be an ϵ -optimal solution, for a fixed $\epsilon > 0$, to the problem on the right-hand side of (3.19). That is,

$$\begin{aligned} \inf_{w \in \mathcal{W}_{[t,r]}|_{\Sigma_{[t,T]}}} \rho_{t,r}^{g_2} \left[\int_t^r c_2(s, X_s^{t,x;\tilde{u}}, \tilde{w}_s) ds + V_2^w(r, X_r^{t,x;\tilde{u}}) \right] + \epsilon \\ \geq \rho_{t,r}^{g_2} \left[\int_t^r c_2(s, X_s^{t,x;\tilde{u}}, \tilde{w}_s) ds + V_2^w(r, X_r^{t,x;\tilde{u}}) \right]. \end{aligned} \quad (3.22)$$

Then, for every $y \in \mathbb{R}^d$, let $\tilde{w} \cdot (y) \in \mathcal{W}_{[t,T]}$ be such that $V_2^w(r, y) + \epsilon \geq V_2^{\tilde{w} \cdot (y)}(t, x)$ and restricted to $\Sigma_{[t,T]}$. Due to the measurable selection theorem, we may assume that the function $y \rightarrow \tilde{w}(y)$ is Borel measurable. Further, suppose that a control function w^0 is defined as follow

$$w_s^0 = \begin{cases} \bar{w}_s, & s \in [t, r) \\ \tilde{w}_s(X_s^{t,x;\bar{u}}), & s \in [r, T]. \end{cases} \quad (3.23)$$

Note that, from the above definition, w^0 is restricted to $\Sigma_{[t,T]}$. Then, using the properties of the monotonicity, translation and time-consistency, we obtain the following

$$\begin{aligned} \rho_{t,r}^{g_2} \left[\int_t^r c_2(s, X_s^{t,x;\bar{u}}, \bar{w}_s) ds + V_2^{\bar{w}}(r, X_r^{t,x;\bar{u}}) \right] \\ \geq \rho_{t,r}^{g_2} \left[\int_t^r c_2(s, X_s^{t,x;\bar{u}}, \bar{w}_s) ds + V_2^{\tilde{w}_s(X_s^{t,x;\bar{u}})}(r, X_r^{t,x;\bar{u}}) - \epsilon \right], \text{ with } \bar{u} = (\hat{v}, \bar{w}) \\ \geq \rho_{t,T}^{g_2} \left[\int_t^T c_2(s, X_s^{t,x;u^0}, \bar{w}_s^0) ds + \Psi_2(X_T^{t,x;u^0}) \right] - \epsilon, \text{ with } u^0 = (\hat{v}, w^0) \\ = V_2^{w^0}(t, x) - \epsilon. \end{aligned} \quad (3.24)$$

If we combine the inequalities from (3.22) and (3.24), then we have

$$\begin{aligned} \inf_{w \in \mathcal{W}_{[t,r]}|_{\Sigma_{[t,T]}}} \rho_{t,r}^{g_2} \left[\int_t^r c_2(s, X_s^{t,x;u}, w_s) ds + V_2^w(r, X_r^{t,x;u}) \right] + \epsilon \geq V_2^{w^0}(t, x) - \epsilon \\ \geq V_2^w(t, x) - \epsilon. \end{aligned} \quad (3.25)$$

Note that, since ϵ is arbitrary, we obtain (3.19). This completes the proof of Proposition 3.5. \square

Then, we have the following results (i.e., Propositions 3.6 and 3.8) that characterize the mapping S in (2.7).

PROPOSITION 3.6. *Suppose that Assumption 1.1 holds and let W be a compact set in \mathbb{R}^d . Let $\hat{v} \cdot \in \mathcal{V}_{[t,T]}$ be given, then the risk-value function $V_2^w(\cdot, \cdot)$ is the viscosity solution of (3.11) with boundary condition $\Psi_2(T, x)$ for $x \in \mathbb{R}^d$ and with $u = (\hat{v}, w)$.*

Proof. Suppose that $\varphi_2 \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$ and assume that $\varphi_2 \geq V_2^w$ on $[0, T] \times \mathbb{R}^d$ and $\max_{(t,x)} [V_2^w(t, x) - \varphi_2(t, x)] = 0$. We consider a point $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ so that $\varphi_2(t_0, x_0) = V_2^w(t_0, x_0)$ (i.e., a local maximum at (t_0, x_0)). Further, for a small $\delta t > 0$, we

consider a constant control $w_s = \alpha$ for $s \in [t_0, t_0 + \delta t]$. Then, from (3.19), we have

$$\begin{aligned} \varphi_2(t_0, x_0) &= V_2^w(t_0, x_0) \\ &\leq \rho_{t_0, t_0 + \delta t}^{g_2} \left[\int_{t_0}^{t_0 + \delta t} c_2(s, X_s^{t_0, x_0; u}, \alpha) ds + V_2^w(t_0 + \delta t, X_{t_0 + \delta t}^{t_0, x_0; u}) \right] \\ &\leq \rho_{t_0, t_0 + \delta t}^{g_2} \left[\int_{t_0}^{t_0 + \delta t} c_2(s, X_s^{t_0, x_0; u}, \alpha) ds + \varphi_2(t_0 + \delta t, X_{t_0 + \delta t}^{t_0, x_0; u}) \right], \text{ with } u = (\hat{v}, \alpha). \end{aligned} \quad (3.26)$$

Using the translation property of $\rho_{t_0, t_0 + \delta t}[\cdot]$, we obtain the following inequality

$$\rho_{t_0, t_0 + \delta t}^{g_2} \left[\int_{t_0}^{t_0 + \delta t} c_2(s, X_s^{t_0, x_0; u}, \alpha) ds + \varphi_2(t_0 + \delta t, X_{t_0 + \delta t}^{t_0, x_0; u}) - \varphi_2(t_0, x_0) \right] \geq 0. \quad (3.27)$$

Notice that $\varphi_2 \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$, then, using Itô integral formula, we can evaluate the difference between $\varphi_2(t_0 + \delta t, X_{t_0 + \delta t}^{t_0, x_0; u})$ and $\varphi_2(t_0, x_0)$ as follow

$$\begin{aligned} \varphi_2(t_0 + \delta t, X_{t_0 + \delta t}^{t_0, x_0; u}) - \varphi_2(t_0, x_0) &= \int_{t_0}^{t_0 + \delta t} \left[\frac{\partial}{\partial t} \varphi_2(s, X_s^{t_0, x_0; u}) + \mathcal{L}_t^{(\hat{v}_s, \alpha)} \varphi_2(s, X_s^{t_0, x_0; u}) \right] ds \\ &\quad + \int_{t_0}^{t_0 + \delta t} D_x \varphi_2(s, X_s^{t_0, x_0; u}) \cdot \sigma(s, X_s^{t_0, x_0; u}, (\hat{v}_s, \alpha)) dB_s. \end{aligned} \quad (3.28)$$

Moreover, if we substitute the above equation into (3.27), then we obtain

$$\begin{aligned} \rho_{t_0, t_0 + \delta t}^{g_2} \left[\int_{t_0}^{t_0 + \delta t} \left[c_2(s, X_s^{t_0, x_0; u}, \alpha) + \frac{\partial}{\partial t} \varphi_2(s, X_s^{t_0, x_0; u}) + \mathcal{L}_t^{(\hat{v}_s, \alpha)} \varphi_2(s, X_s^{t_0, x_0; u}) \right] ds \right. \\ \left. + \int_{t_0}^{t_0 + \delta t} D_x \varphi_2(s, X_s^{t_0, x_0; u}) \cdot \sigma(s, X_s^{t_0, x_0; u}, (\hat{v}_s, \alpha)) dB_s \right] \geq 0, \end{aligned} \quad (3.29)$$

which amounts to solving the following BSDE

$$\begin{aligned} Y_{t_0}^{2; t_0, x_0; u} &= \int_{t_0}^{t_0 + \delta t} \left[c_2(s, X_s^{t_0, x_0; u}, \alpha) + \frac{\partial}{\partial t} \varphi_2(s, X_s^{t_0, x_0; u}) + \mathcal{L}_t^{(\hat{v}, \alpha)} \varphi_2(s, X_s^{t_0, x_0; u}) \right] ds \\ &\quad + \int_{t_0}^{t_0 + \delta t} D_x \varphi_2(s, X_s^{t_0, x_0; u}) \cdot \sigma(s, X_s^{t_0, x_0; u}, (\hat{v}_s, \alpha)) dB_s \\ &\quad + \int_{t_0}^{t_0 + \delta t} g_2(s, Z_s^{2; t_0, x_0; u}) ds - \int_{t_0}^{t_0 + \delta t} Z_s^{2; t_0, x_0; u} dB_s. \end{aligned} \quad (3.30)$$

From Lemma 1.2, the above BSDE admits unique solutions, i.e.,

$$Z_s^{2; t_0, x_0; u} = D_x \varphi_2(s, X_s^{t_0, x_0; u}) \cdot \sigma(s, X_s^{t_0, x_0; u}, (\hat{v}_s, \alpha)), \quad t_0 \leq s \leq t_0 + \delta t$$

and

$$\begin{aligned} Y_{t_0}^{2; t_0, x_0; u} &= \int_{t_0}^{t_0 + \delta t} \left[c_2(s, X_s^{t_0, x_0; u}, \alpha) + \frac{\partial}{\partial t} \varphi_2(s, X_s^{t_0, x_0; u}) + \mathcal{L}_t^{(\hat{v}_s, \alpha)} \varphi_2(s, X_s^{t_0, x_0; u}) \right. \\ &\quad \left. + g_2(s, D_x \varphi_2(s, X_s^{t_0, x_0; u}) \cdot \sigma(s, X_s^{t_0, x_0; u}, (\hat{v}_s, \alpha))) \right] ds. \end{aligned}$$

Further, if we substitute the above results in (3.29), we obtain

$$\int_{t_0}^{t_0+\delta t} \left[c_2(s, X_s^{t_0, x_0; u}, \alpha) + \frac{\partial}{\partial t} \varphi_2(s, X_s^{t_0, x_0; u}) + \mathcal{L}_t^{(\hat{v}, \alpha)} \varphi_2(s, X_s^{t_0, x_0; u}) \right. \\ \left. + g_2(s, D_x \varphi_2(s, X_s^{t_0, x_0; u}) \cdot \sigma(s, X_s^{t_0, x_0; u}, (\hat{v}_s, \alpha))) \right] ds \geq 0. \quad (3.31)$$

Then, dividing the above equation by δt and letting $\delta t \rightarrow 0$, we obtain

$$c_2(t_0, x_0, \alpha) + \frac{\partial}{\partial t} \varphi_2(t_0, x_0) + \mathcal{L}_t^{(\hat{v}, \alpha)} \varphi_2(t_0, x_0) \\ + g_2(t_0, D_x \varphi_2(t_0, x_0) \cdot \sigma(t_0, x_0, (\hat{v}_{t_0}, \alpha))) \geq 0.$$

Note that, since $\alpha \in W$ is arbitrary, we can rewrite the above condition as follow

$$\frac{\partial}{\partial t_0} \varphi_2(t_0, x_0) + \min_{\alpha \in W} \left\{ c_2(t_0, x_0, \alpha) + \mathcal{L}_t^{(\hat{v}, \alpha)} \varphi_2(t_0, x_0) \right. \\ \left. + g_2(t_0, D_x \varphi_2(t_0, x_0) \cdot \sigma(t_0, x_0, (\hat{v}_{t_0}, \alpha))) \right\} \geq 0, \quad (3.32)$$

which attains its minimum in W (which is a compact set in \mathbb{R}^d). Thus, $V_2^w(\cdot, \cdot)$ is a viscosity subsolution of (3.36), with boundary condition $\varphi_2(T, x) = \Psi_2(T, x)$.

On the other hand, suppose that $\varphi_2 \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$ and assume that $\varphi_2 \leq V_2^w$ on $[0, T] \times \mathbb{R}^d$ and $\min_{(t, x)} [V_2^w(t, x) - \varphi_2(t, x)] = 0$. Then, we consider a point $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ so that $\varphi_2(t_0, x_0) = V_2^w(t_0, x_0)$ (i.e., a local minimum at (t_0, x_0)). Further, for a small $\delta t > 0$, Let \tilde{w}_s , which is restricted to $\Sigma_{[t_0, t_0+\delta t]}$, be an $\epsilon \delta t$ -optimal control for (3.19) at (t_0, x_0) . Then, proceeding in this way as (3.31), we obtain the following

$$\int_{t_0}^{t_0+\delta t} \left[c_2(s, X_s^{t_0, x_0; u}, \tilde{w}_s) + \frac{\partial}{\partial t} \varphi_2(s, X_s^{t_0, x_0; u}) + \mathcal{L}_t^{(\hat{v}_s, \tilde{w}_s)} \varphi_2(s, X_s^{t_0, x_0; u}) \right. \\ \left. + g_2(s, D_x \varphi_2(s, X_s^{t_0, x_0; u}) \cdot \sigma(s, X_s^{t_0, x_0; u}, (\hat{v}_s, \tilde{w}_s))) \right] ds \leq \epsilon \delta t, \text{ with } u = (\hat{v}_s, \tilde{w}_s). \quad (3.33)$$

As a result of this, we also obtain the following

$$\int_{t_0}^{t_0+\delta t} \min_{\alpha \in W} \left\{ c_2(s, X_s^{t_0, x_0; u}, \alpha) + \frac{\partial}{\partial t} \varphi_2(s, X_s^{t_0, x_0; u}) + \mathcal{L}_t^{(\hat{v}, \alpha)} \varphi_2(s, X_s^{t_0, x_0; u}) \right. \\ \left. + g_2(s, D_x \varphi_2(s, X_s^{t_0, x_0; u}) \cdot \sigma(s, X_s^{t_0, x_0; u}, (\hat{v}_s, \alpha))) \right\} ds \leq \epsilon \delta t. \quad (3.34)$$

Note that the mapping

$$(s, x, \alpha) \rightarrow \left[c_2(s, x, \alpha) + \frac{\partial}{\partial t} \varphi_2(t, x) + \mathcal{L}_t^{(\hat{v}_t, \alpha)} \varphi_2(t, x) \right. \\ \left. + g_2(t, D_x \varphi_2(t, x) \cdot \sigma(t, x, (\hat{v}_t, \alpha))) \right]$$

is continuous and, since W is compact, then $s \rightarrow X_s^{t_0, x_0; u}$ is also continuous. As a result, the expression under the integral in (3.34) is continuous. Further, if we divide both sides of

(3.34) by δt and letting $\delta t \rightarrow 0$, then we obtain the following

$$\begin{aligned} \frac{\partial}{\partial t_0} \varphi_2(t_0, x_0) + \min_{\alpha \in W} \left\{ c_2(t_0, x_0, \alpha) + \mathcal{L}_t^{(\hat{v}, \alpha)} \varphi_2(t_0, x_0) \right. \\ \left. + g_2(t_0, D_x \varphi_2(t_0, x_0) \cdot \sigma(t_0, x_0, (\hat{v}_{t_0}, \alpha))) \right\} \leq \epsilon. \end{aligned} \quad (3.35)$$

Notice that, since ϵ is arbitrary, we conclude that $V_2^w(\cdot, \cdot)$ is a viscosity supersolution of (3.18), with boundary condition $\varphi_2(T, x) = \Psi_2(T, x)$. This completes the proof of Proposition 3.6. \square

REMARK 3.7. *Note that if $V_2^w \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$, then such a solution also satisfies (3.18) with boundary condition $V_2^w(T, x) = \Psi_2(T, x)$. Furthermore, using the verification theorem, one can also identify V_2^w as the optimal value function.*

PROPOSITION 3.8. *Suppose that Proposition 3.6 holds and let $\varphi_2 \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$ satisfy (3.11) with $\varphi_2(T, x) = \Psi_2(T, x)$ for $x \in \mathbb{R}^d$. Then, $\varphi_2(t, x) \leq V_2^w(t, x)$ for any control $w. \in \mathcal{W}_{[t, T]}$ with restriction to $\Sigma_{[t, T]}$ and for all $(t, x) \in [0, T] \times \mathbb{R}^d$. Furthermore, if an admissible control process $\hat{w}. \in \mathcal{W}_{[t, T]}$ exists, for almost all $(s, \Omega) \in [0, T] \times \Omega$, together with the corresponding solution $X_s^{t, x; \hat{w}.}$, with $\hat{u}_s = (\hat{v}_s, \hat{w}_s)$, and satisfies*

$$\begin{aligned} \hat{w}_s \in \arg \inf_{w. \in \mathcal{W}_{[t, T]} \big|_{\Sigma_{[t, T]}}} \left\{ c_2(s, X_s^{t, x; u}, w_s) + \mathcal{L}_s^{(v, w)} \varphi_2(s, X_s^{t, x; u}) \right. \\ \left. + g_2(s, D_x \varphi_2(s, X_s^{t, x; u}) \cdot \sigma(s, X_s^{t, x; u}, (\hat{v}_s, w_s))) \right\} \end{aligned} \quad (3.36)$$

$\underbrace{\hspace{15em}}_{\triangleq S(\hat{v}) \text{ with } S: \mathcal{V}_{[t, T]} \Rightarrow \mathcal{W}_{[t, T]}}$

Then, $\varphi_2(t, x) = V_2^{\hat{w}.}(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

Proof. Assume that $(t, x) \in [0, T] \times \mathbb{R}^d$ is fixed. For any $w. \in \mathcal{W}_{[t, T]}$, restricted to $\Sigma_{[t, T]}$, we consider a process $\kappa(s, X_s^{t, x; u})$, with $u = (\hat{v}, w)$, for $s \in [t, T]$. Then, using Itô integral formula, we can evaluate the difference between $\kappa(T, X_T^{t, x; u})$ and $\kappa(t, x)$ as follow⁵

$$\begin{aligned} \kappa(T, X_T^{t, x; u}) - \kappa(t, x) = \int_t^T \left[\frac{\partial}{\partial t} \kappa(s, X_s^{t, x; u}) + \mathcal{L}_t^{(\hat{v}_s, w_s)} \kappa(s, X_s^{t, x; u}) \right] ds \\ + \int_t^T D_x \kappa(s, X_s^{t, x; u}) \cdot \sigma(s, X_s^{t, x; u}, (\hat{v}_s, w_s)) dB_s. \end{aligned} \quad (3.37)$$

Using (3.11), we further obtain the following

$$\begin{aligned} \frac{\partial}{\partial t} \kappa(s, X_s^{t, x; u}) + \mathcal{L}_t^{(\hat{v}_s, w_s)} \kappa(s, X_s^{t, x; u}) + c_2(s, X_s^{t, x; u}, w_s) \\ + g_2(s, D_x \varphi_2(s, X_s^{t, x; u}) \cdot \sigma(s, X_s^{t, x; u}, (\hat{v}_s, w_s))) \geq 0. \end{aligned} \quad (3.38)$$

⁵Notice that $\kappa(t, x) \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$.

Furthermore, if we combine (3.37) and (3.38), then we obtain

$$\begin{aligned} \kappa(t, x) \leq & \Psi_2(T, X_T^{t,x;u}) + \int_t^T \left\{ c_2(s, X_s^{t,x;u}, w_s) ds \right. \\ & + g_2(s, D_x \kappa(s, X_s^{t,x;u}) \cdot \sigma(s, X_s^{t,x;u}, (\hat{v}_s, w_s))) \Big\} ds \\ & - \int_t^T D_x \kappa(s, X_s^{t,x;u}) \cdot \sigma(s, X_s^{t,x;u}, (\hat{v}_s, w_s)) dB_s. \end{aligned} \quad (3.39)$$

Define $Z_s^{2;t,x;u} = D_x \kappa(s, X_s^{t,x;u}) \cdot \sigma(s, X_s^{t,x;u}, (\hat{v}_s, w_s))$, for $s \in [t, T]$, then $\kappa(t, x) \leq Y_t^{2,t,x;u}$ follows, where $(Y^{2;t,x;u}, Z^{2;t,x;u})$ is a solution to BSDE in (2.9). As a result of this, we have

$$\kappa(t, x) \leq V_2^w(t, x).$$

Moreover, if there exists at least one \hat{w} satisfying (3.36). Then, for $w = \hat{w}$, the inequality in (3.39) becomes an equality (i.e., $\kappa(t, x) = V_2^{\hat{w}}(t, x)$). Note that the corresponding pathwise solution $X_s^{t,x;\hat{u}}$, with $\hat{u} = (\hat{v}, \hat{w})$ and $\hat{w} = S(\hat{v})$, is progressively measurable, since the control process $\hat{w} \in \mathcal{W}_{[t,T]}$ is restricted to $\Sigma_{[t,T]}$. This completes the proof of Proposition 3.8. \square

3.2. On the risk-averse optimality condition for the leader. In this subsection, we provide an optimality condition on the strategy of the *leader* in (2.6) (cf. equations 2.2 and (2.15)), when the risk-averse strategy for the *follower* satisfies the optimality condition of Proposition 3.8.

PROPOSITION 3.9. *Suppose that Proposition 3.8 holds. Then, for any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $r \in [t, T]$, the risk-value function w.r.t. the leader is given by*

$$V_1^v(t, x) = \inf_{v \in \mathcal{V}_{[t,r]} | \Sigma_{[t,T]}} \rho_{t,r}^{g_1} \left[\int_t^r c_1(s, X_s^{t,x;u}, v_s) ds + V_1^v(r, X_r^{t,x;u}) \right] \quad (3.40)$$

Furthermore, if V is a compact set in \mathbb{R}^d , then $V_1^v(\cdot, \cdot)$ is the viscosity solution of (3.10) with boundary condition $\Psi_1(T, x)$ for $x \in \mathbb{R}^d$.

Proof. Note that if Proposition 3.8 holds, then, for any control process $v \in \mathcal{V}_{[t,T]}$, there exists at least one $w \in S(v) \subset \mathcal{W}_{[t,T]}$ such that S satisfies the definition of the mapping in (3.36). As a result, one can prove in the same way as Proposition 3.6. \square

PROPOSITION 3.10. *Suppose that Proposition 3.9 holds and let $\varphi_1 \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$ satisfy (3.10) with boundary condition $\varphi_1(T, x) = \Psi_1(T, x)$ for $x \in \mathbb{R}^d$. Then, $\varphi_1(t, x) \leq V_1^v(t, x)$ for any control process $v \in \mathcal{V}_{[t,T]}$ with restriction to $\Sigma_{[t,T]}$ and for all $(t, x) \in [0, T] \times \mathbb{R}^d$. Furthermore, if an admissible control process $v^* \in \mathcal{V}_{[t,T]}$ exists, for almost all $(s, \Omega) \in [0, T] \times \Omega$, together with the corresponding solution $X_s^{t,x;u^*}$, with $u_s^* = (v_s^*, S(v_s^*))$, and satisfies*

$$\begin{aligned} v_s^* \in & \arg \inf_{v \in \mathcal{V}_{[t,T]} | \Sigma_{[t,T]}} \left\{ c_1(s, X_s^{t,x;u}, v_s) + \mathcal{L}_s^{(v, S(v))} \varphi_1(s, X_s^{t,x;u}) \right. \\ & \left. + g_1(s, D_x \varphi_1(s, X_s^{t,x;u}) \cdot \sigma(s, X_s^{t,x;u}, (v_s, S(v_s)))) \right\}. \end{aligned} \quad (3.41)$$

Then, $\varphi_1(t, x) = V_1^{v^*}(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

Proof. The proof is similar to that of Proposition 3.8, except that we require a unique solution set $\{X_s^{t,x;u}, (Y_s^{1;t,x;u}, Z_s^{1;t,x;u}), (Y_s^{2;t,x;u}, Z_s^{2;t,x;u})\}$ for the FBSDEs in (2.1), (2.8) and (2.9) on $(\Omega, \mathcal{F}, P, \mathcal{F}^t)$ for every initial condition $(t, x) \in [0, T] \times \mathbb{R}^d$. \square

REMARK 3.11. *Observe that the condition in (3.41) requires the follower to respond optimally to the risk-averse strategy of the leader, where such a correspondence is implicitly embedded via the mapping S (cf. equation (3.36)). Here it is worth remarking that the follower is involved not only in minimizing his own accumulated risk-cost (in response to the risk-averse strategy of the leader) but also in minimizing that of the leader. Furthermore, such leader-follower optimal risk-averse strategy pairs (v^*, w^*) solve, in the sense of viscosity, the associated risk-averse dynamic programming equations of (3.19) and (3.41).*

4. Further remarks. In this section, we further comment on the implication of our result in assessing the influence of the *leader's* risk-averse satisfaction on the risk-averseness of the *follower* in relation to the direction of *leader-follower* information flow. Note that the statement of Proposition 3.8 is implicitly accounted in Proposition 3.10. That is, for $s \in [t, T]$, the risk-averseness of the *follower*, with restriction to $\Sigma_{[t,T]}$,

$$w_s^* \in \arg \inf_{w \in \mathcal{W}_{[t,T]} | \Sigma_{[t,T]}} \left\{ c_2(s, X_s^{t,x;u}, w_s) + \mathcal{L}_s^{(v,w)} \varphi_2(s, X_s^{t,x;u}) + g_2(s, D_x \varphi_2(s, X_s^{t,x;u}) \cdot \sigma(s, X_s^{t,x;u}, (v_s, w_s))) \right\}$$

is a subproblem in (3.41). On the other hand, the risk-averse strategy of the *leader*

$$v_s^* \in \arg \inf_{v \in \mathcal{V}_{[t,T]} | \Sigma_{[t,T]}} \left\{ c_1(s, X_s^{t,x;u}, v_s) + \mathcal{L}_s^{(v,S(v))} \varphi_1(s, X_s^{0,x;u}) + g_1(s, D_x \varphi_1(s, X_s^{t,x;u}) \cdot \sigma(s, X_s^{t,x;u}, (v_s, S(v_s)))) \right\},$$

that is implicitly conditioned by the *leader's* risk-averse satisfaction and that of the *follower's* risk-averse strategy $w \in S(v)$. As a result of this, the *follower* is involved not only in minimizing his own accumulated risk-cost (in response to the risk-averse strategy of the *leader*) but also in minimizing that of the *leader's* accumulated risk-cost.⁶ Hence, such an inherent interaction, due to the *nature of the problem*, constitutes a constrained information flow between the *leader* and that of the *follower*, in which the *follower* is required to respond optimally, in the sense of *best-response correspondence* to the risk-averse strategy of the *leader* (cf. Footnote 4).

REMARK 4.1 (*Additional comment on model uncertainty*). Finally, it is worth remarking that the issue of risk-averseness under model uncertainty, when the follower is allowed to take

⁶Notice that the *follower's* optimal risk-averse strategy is also given by

$$w_s^* \in \arg \inf_{w \in S_b(v^*) | \Sigma_{[t,T]}} \left\{ c_1(s, X_s^{t,x;u}, v_s^*) + \mathcal{L}_s^{(v^*,w)} \varphi_1(s, X_s^{t,x;u}) + g_1(s, D_x \varphi_1(s, X_s^{t,x;u}) \cdot \sigma(s, X_s^{t,x;u}, (v_s^*, w_s))) \right\}, \text{ with } u = (v^*, w),$$

where the set of all *best-responses* of the *follower* $S_b(v^*)$ is nonempty and

$$S_b(v^*) \subseteq \{w \in \mathcal{W}_{[t,T]} | v^* \in \mathcal{V}_{[t,T]} \text{ \& } \rho_{t,T}^{g_2}[\xi_{t,T}^2(v^*, w)] < \rho_{t,T}^{g_2}[\xi_{t,T}^2(v^*, \hat{w})], \forall \hat{w} \in \mathcal{W}_{[t,T]}\}.$$

into account alternative models that are statistically difficult to distinguish from (1.1), can be modeled as an “inf-sup” optimization problem with uncertainty aversion. For example, if we replace B_t in (1.1) with

$$B_t = \widehat{B}_t + \int_0^t h_s ds,$$

where $h \in C_b^{1,2}([0, T]; \mathbb{R}^d)$ is a measurable function and \widehat{B}_t is a d -dimensional brownian motion. Then, the process h can be used as a device to transform \mathbb{P} into another probability distribution \mathbb{Q} which is mutually absolutely continuous w.r.t. \mathbb{P} on (Ω, \mathcal{F}) (e.g., see Girsavov [12] for additional discussions on transforming stochastic processes). Furthermore, if we specify the model uncertainty in terms of the relative entropy between \mathbb{Q} and \mathbb{P} as a single constraint on the entire path of perturbation. Then, considering h as an adversarial control process in a two-player zero-sum stochastic differential game (in the sense of Elliot and Kalton [7]), we can reformulate (2.11) as an “inf-sup” problem with uncertainty-aversion between the follower and that of the adversary (e.g., see [13] or [3] for related discussions, but in a different context).

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